SOME PROPERTIES OF A NEW SUBCLASS OF BAZILEVIC FUNCTIONS DEFINED BY CATAS *et al.* DIFFERENTIAL OPERATOR

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Abstract: Complex analysis is simply the culmination of a thorough study of fundamental concepts of complex differentiation and integration, and has a flavour not found in the real domain. It can effectively used for constructing solutions to the Laplace equation on complicated planar domains which can be found in a broad range of physical problems, such as fluid mechanics, thermodynamics, aerodynamics, electrostatics and elasticity. However, in the present paper, the authors introduced and studied a new subclass $M_{\lambda,l}^{\alpha,m}(\gamma,\beta)$ of Bazilevic type in the open unit disk where coefficient inequalities, distortion theorems, modified Hadamard products among others for functions belonging to the aforementioned subclass were obtained while several interesting corollaries follow as simple consequences.

Keywords: Analytic function, Catas operator, coefficient inequalities, Hadamard product

Introduction

Suppose that A_j denote the class of functions of the form;

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \quad (j \in N = \{1,2,...\}), \tag{1}$$

analytic in the open unit disk $U = \{z : z \in C \quad and \quad |z| < 1\}$.

Catas *et al*. (2008) introduced and studied the differential operator $I^m(\lambda,l)f(z)$ for $f(z) \in A_j(p)$ such that;

$$I^0 f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k = f(z)$$

$$I^1 f(z) = \left(I^0 f(z)\right)\left(\frac{1-\lambda+l}{1+l}\right) + \left(I^0 f(z)\right)' \frac{\lambda z}{1+l},$$

$$I^2 f(z) = \left(I^1 f(z)\right)\left(\frac{1-\lambda+l}{1+l}\right) + \left(I^1(\lambda,l) f(z)\right)' \frac{\lambda z}{1+l},$$

$$\vdots$$

$$I^m(\lambda,l) f(z) = \left(I^{m-1}(\lambda,l) f(z)\right)\left(\frac{1-\lambda+l}{1+l}\right) + \left(I^{m-1}(\lambda,l) f(z)\right)' \frac{\lambda z}{1+l}$$

$$(\lambda > 0, l \geq 0, m \in N \cup \{0\}, j \in N \text{ and } z \in U). \tag{2}$$

From (1), one can write that

$$f(z)^\alpha = z^\alpha - \sum_{k=j+1}^{\infty} a_k(\alpha) z^{\alpha+k-1} \tag{3}$$

$\alpha > 0 \quad (\alpha \text{ is real and } z \in U)$. Using (2) in (3), then

$$I^m(\lambda,l) f(z)^\alpha = \left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m z^\alpha - \sum_{k=j+1}^{\infty} \left(\frac{1+\lambda(\alpha+k-2)+l}{1+l}\right)^m a_k(\alpha) z^{\alpha+k-1}. \tag{4}$$

For detailed one can see Hamzat and Olayiwola, 2015; Oladipo and Breaz, 2013.
Now (4) can be written as;

$$\frac{I^m(\lambda,l) f(z)^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^m z^\alpha} = 1 - \sum_{k=j+1}^{\infty} \left(\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right)^m a_k(\alpha) z^{k-1}$$

Or as;

$$\frac{(1+l)^m I^m(\lambda,l)f(z)^\alpha}{(1+\lambda(\alpha-1)+l)^m z^\alpha}=1-\sum_{k=j+1}^{\infty}\left(\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right)^m a_k(\alpha)z^{k-1} \quad (5)$$

$$(\alpha>0,\lambda>0,l\geq 0,m\in N\cup\{0\}, j\in N \; and \; z\in U).$$

Remark

It is observed that if $\lambda=1$ and $l=0$ in (5), then the expression readily becomes the famous class of Bazilevic function studied by different authors (Opoola, 1994; Oladipo and Breaz, 2013).

With the aid of (5), we give the following definition which shall be necessary for the sake of the present investigation.

Definition: Let $f(z)^\alpha$ be given by (3), then $f(z)^\alpha \in M_{\lambda,l}^{\alpha,m}(\gamma,\beta)$ if and only if

$$\left|\frac{\dfrac{(1+l)^m I^m(\lambda,l)f(z)^\alpha}{(1+\lambda(\alpha-1)+l)^m z^\alpha}-1}{\dfrac{(1+l)^m I^m(\lambda,l)f(z)^\alpha}{(1+\lambda(\alpha-1)+l)^m z^\alpha}+1-2\gamma}\right|<\beta \quad (6)$$

$$(\alpha>0,\lambda\geq 0,l\geq 0,m\in N\cup\{0\},0\leq\gamma<1,0<\beta\leq 1 \;\; and \;\; z\in U).$$

Theorem 2.1: Let the function $f(z)^\alpha$ be defined by (3).Then, $f(z)^\alpha \in M_{\lambda,l}^{\alpha,m}(\gamma,\beta)$ if and only if;

$$\sum_{k=j+1}^{\infty}(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m a_k(\alpha)\leq 2\beta(1-\gamma)$$

$$(0\leq\gamma<1,\alpha>0 \;(\alpha \;\; is \;\; real),0<\beta\leq 1,l\geq 0,m\in N_0 \quad z\in U). \quad (7)$$

Proof: Assuming that the inequality (7) holds true. Then, for $|z|=r<1$, we show that;

$$\left|\frac{(1+l)^m I^m(\lambda,l)f(z)^\alpha}{(1+\lambda(\alpha-1)+l)^m z^\alpha}-1\right|-\beta\left|\frac{(1+l)^m I^m(\lambda,l)f(z)^\alpha}{(1+\lambda(\alpha-1)+l)^m z^\alpha}+1-2\gamma\right|<0$$

$$\left|-\sum_{k=j+1}^{\infty}\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m a_k(\alpha)z^{k-1}\right|-\beta\left|-\sum_{k=j+1}^{\infty}\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m a_k(\alpha)z^{k-1}+2(1-\gamma)\right|<0.$$

$$\left|-\sum_{k=j+1}^{\infty}\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m a_k(\alpha)z^{k-1}\right|-\beta\left|2(1-\gamma)-\sum_{k=j+1}^{\infty}\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m a_k(\alpha)z^{k-1}\right|$$

$$\leq\sum_{k=j+1}^{\infty}\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m a_k(\alpha)-\beta\left[2(1-\gamma)-\sum_{k=j+1}^{\infty}\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha+k-2)+l}\right]^m a_k(\alpha)\right],$$

$$\leq\sum_{k=j+1}^{\infty}(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m a_k(\alpha)-2\beta(1-\gamma)\leq 0.$$

Therefore, $f(z)^\alpha \in M_{\lambda,l}^{\alpha,m}(\gamma,\beta)$.

Conversely, let $f(z)^\alpha \in M_{\lambda,l}^{\alpha,m}(\gamma,\beta)$. Then;

$$\left|\frac{\dfrac{(1+l)^m I^m(\lambda,l)f(z)^\alpha}{(1+\lambda(\alpha-1)+l)z^\alpha}-1}{\dfrac{(1+l)^m I^m(\lambda,l)f(z)^\alpha}{(1+\lambda(\alpha-1)+l)z^\alpha}+1-2\gamma}\right|=\left|\frac{-\sum_{k=j+1}^{\infty}\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m a_k(\alpha)z^{k-1}}{2(1-\gamma)-\sum_{k=j+1}^{\infty}\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m a_k(\alpha)z^{k-1}}\right|<\beta.$$

Since $\mathbf{Re}(z)<|z|$ for all z, the following inequality;

$$\mathbf{Re}\left\{\frac{\sum_{k=j+1}^{\infty}\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^{m}a_k(\alpha)z^{k-1}}{2(1-\gamma)-\sum_{k=j+1}^{\infty}\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^{m}a_k(\alpha)z^{k-1}}\right\}<\beta \quad (z\in U), \tag{8}$$

is obtained. By choosing the values of z on the real axis so that $\frac{(1+l)^m I^m(\lambda,l)f(z)^\alpha}{(1+\lambda(\alpha-1)+l)z^\alpha}$ is real and clearing the denominator in (8) as $z\to 1^-$ through real values, then;

$$\sum_{k=j+1}^{\infty}\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^{m}a_k(\alpha)\le 2\beta(1-\gamma)-\sum_{k=j+1}^{\infty}\beta\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^{m}a_k(\alpha),$$

which readily gives the inequality (7) and thus the proof of theorem 2.1.

Corollary 2.2. Let the function $f(z)^\alpha$ defined by (3) be in the class $M_{\lambda,l}^{\alpha,m}(\gamma,\beta)$. Then

$$a_k(\alpha)\le\frac{2\beta(1-\gamma)(1+\lambda(\alpha-1)+l)^m}{(1+\beta)(1+\lambda(\alpha+k-2)+l)^m}\quad (k\ge j+1, j\in N).$$

This result is sharp for function $f(z)^\alpha$ given by;

$$f(z)^\alpha = z^\alpha-\frac{2\beta(1-\gamma)(1+\lambda(\alpha-1)+l)^m}{(1+\beta)(1+\lambda(\alpha+k-2)+l)^m}\mathbf{z}^{\alpha+\mathbf{k}-\mathbf{1}}\quad (k\ge j+1, j\in N).$$

Corollary 2.3. Let the function $f(z)^\alpha$ defined by (3). Then $f(z)^\alpha\in M_{1,0}^{1,m}(\gamma,\beta)$ if and only if,

$$\sum_{k=j+1}^{\infty}(1+\beta)k^m a_k(1)\le 2\beta(1-\gamma) \tag{9}$$

$$(0\le\gamma<1,\ \alpha>0,\ m\in N_0,\ \lambda\ge 0,\ l\ge 0,\ 0<\beta\le 1,\ z\in U).$$

Theorem 2.4. For $\alpha>0,\ 0<\beta\le 1,\ 0\le\gamma<1,\ l\ge 0,\ \lambda\ge 0,\ m\in N_0$ and $z\in U$, $M_{\lambda,l}^{\alpha,m+1}(\gamma,\beta)\subseteq M_{\lambda,l}^{\alpha,m}(\delta,\beta)$, where $\delta=\gamma$.

Proof: Let the function $f(z)^\alpha$ defined by (3) be in the class $M_{\lambda,l}^{\alpha,m+1}(\gamma,\beta)$, then by theorem 2.1,

$$\sum_{k=j+1}^{\infty}(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^{m+1}a_k(\alpha)\le 2\beta(1-\gamma). \tag{10}$$

Then, one shows that;

$$\sum_{k=j+1}^{\infty}(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^{m}a_k(\alpha)\le 2\beta(1-\delta). \tag{11}$$

With reference to (10), (11) it will be possible if;

$$\frac{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^{m}a_k(\alpha)}{2\beta(1-\delta)}\le\frac{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^{m+1}a_k(\alpha)}{2\beta(1-\gamma)},\ k\ge j+1, j\in N.$$

That is $1-\gamma\le 1-\delta$ which yield desired result and hence the proof.

Distortion Theorems

Theorem 2.5 Let the function $f(z)^\alpha$ defined by (3) be in the class $M_{\lambda,l}^{\alpha,m}(\gamma,\beta)$. Then $|z|=r<1$,

$$\left|\frac{(1+l)^i I^i(\lambda,l)f(z)^\alpha}{(1+\lambda(\alpha-1)+l)^i z^\alpha}\right|\ge 1-\frac{2\beta(1-\gamma)}{(1+\beta)\left[\frac{1+\lambda(\alpha+j-1)+l}{1+\lambda(\alpha-1)+l}\right]^{m-i}}r^j, \tag{12}$$

$$\left|\frac{(1+l)^i I^i(\lambda,l) f(z)^\alpha}{(1+\lambda(\alpha-1)+l)^i z^\alpha}\right| \le 1+\frac{2\beta(1-\gamma)}{(1+\beta)\left[\frac{1+\lambda(\alpha+j-1)+l}{1+\lambda(\alpha-1)+l}\right]^{m-i}} r^j . \quad (13)$$

For $z \in U$ and $0 \le i \le m$. The above inequalities are attained for the function $f(z)^\alpha$ given by;

$$f(z)^\alpha = z^\alpha - \frac{2\beta(1\text{-}\gamma)}{(1+\beta)\left[\frac{1+\lambda(\alpha+j\text{-}1)+l}{1+\lambda(\alpha\text{-}1)+l}\right]^m} z^j \ (z=\pm r). \quad (14)$$

Proof: Note that $f(z)^\alpha \in M_{\lambda,l}^{\alpha,m}(\gamma,\beta)$ if and only if;

$$\frac{(1+l)^i I^i(\lambda,l) f(z)^\alpha}{[1+\lambda(\alpha-1)+l]^i z^\alpha} \in M_{\lambda,l}^{\alpha,m-i}(\gamma,\beta)$$

and that;

$$\frac{(1+l)^i I^i(\lambda,l) f(z)^\alpha}{(1+\lambda(\alpha-1)+l)^i z^\alpha} = 1-\sum_{k=j+1}^{\infty}\left[\frac{1+\lambda(\alpha+k-2)l}{1+\lambda(\alpha-1)+l}\right]^i a_k(\alpha) z^{k-1}. \quad (15)$$

From theorem 2.1, we have that;

$$(1+\beta)\left[\frac{1+\lambda(\alpha+j-1)+l}{1+\lambda(\alpha-1)+l}\right]^{m-i}\sum_{k=j+1}^{\infty}\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^i a_k(\alpha) \le \sum_{k=j+1}^{\infty}\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m a_k(\alpha) \le 2\beta(1-\gamma)$$

It implies that;

$$\sum_{k=j+1}^{\infty}\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^i a_k(\alpha) \le \frac{2\beta(1-\gamma)}{(1+\beta)\left[\frac{1+\lambda(\alpha+j-1)+l}{1+\lambda(\alpha-1)+l}\right]^{m-i}}. \quad (16)$$

The inequalities (12) and (13) of theorem 2.5 would now follow immediately from (15) and (16) and this ends the proof of theorem 2.5.

Corollary 2.6: Let the function $f(z)^\alpha$ defined by (3) be in the class $M_{\lambda,l}^{\alpha,m}(\gamma,\beta)$. Then for $|z| = r < 1$

$$\left|f(z)^\alpha\right| \ge r^\alpha - \frac{2\beta(1-\gamma)}{(1+\beta)\left[\frac{1+\lambda(\alpha+j-1)+l}{1+\lambda(\alpha-1)+l}\right]^m} r^{\alpha+j} \quad (17)$$

and

$$\left|f(z)^\alpha\right| \le r^\alpha + \frac{2\beta(1-\gamma)}{(1+\beta)\left[\frac{1+\lambda(\alpha+j-1)+l}{1+\lambda(\alpha-1)+l}\right]^m} r^{\alpha+j}. \quad (18)$$

The inequalities in (17) and (18) are attained for function $f(z)^\alpha$ given by (14).

Proof: Letting $i = 0$ in the theorem 2.5, our results in (17) and (18) follow immediately.

Corollary 2.7: Let the function $f(z)^\alpha$ defined by (3) be in the class $M_{\lambda,l}^{\alpha,m}(\gamma,\beta)$. Then for $|z| = r < 1$

$$\left|(f(z)^\alpha)'\right| \ge \alpha r^{\alpha-1} - \frac{2\beta(1-\gamma)(\alpha+j)}{(1+\beta)\left[\frac{1+\lambda(\alpha+j-1)+l}{1+\lambda(\alpha-1)+l}\right]^m} r^{\alpha+j-1} \quad (19)$$

and

$$\left|(f(z)^\alpha)'\right| \le \alpha r^{\alpha-1} - \frac{2(\alpha+j)\beta(1-\gamma)}{(1+\beta)\left[\frac{1+\lambda(\alpha+j-1)+l}{1+\lambda(\alpha-1)+l}\right]^m} r^{\alpha+j-1} \quad (20)$$

The inequalities in (19) and (20) are attained for function $f(z)^\alpha$ given by (14).

Corollary 2.8: Let the function $f(z)^{\alpha}$ defined by (3) be in the class $M_{1,0}^{\alpha,m}(\gamma,\beta)$. Then for $|z| = r < 1$

$$\left|\frac{I^i(1,0)f(z)^{\alpha}}{\alpha^i z^{\alpha}}\right| \geq 1 - \frac{2\beta(1-\gamma)}{(1+\beta)\left[\frac{\alpha+j}{\alpha}\right]^{m-i}} r^j \tag{21}$$

and

$$\left|\frac{I^i(1,0)f(z)^{\alpha}}{\alpha^i z^{\alpha}}\right| \leq 1 + \frac{2\beta(1-\gamma)}{(1+\beta)\left[\frac{\alpha+j}{\alpha}\right]^{m-i}} r^j \tag{22}$$

Corollary 2.9: Let the function $f(z)^{\alpha}$ defined by (3) be in the class $M_{1,0}^{\alpha,m}(\gamma,\beta)$. Then for $|z| = r < 1$

$$\left|f(z)^{\alpha}\right| \geq r^{\alpha} - \frac{2\beta(1-\gamma)}{(1+\beta)\left[\frac{\alpha+j}{\alpha}\right]^{m-i}} r^{\alpha+j} \tag{23}$$

and

$$\left|f(z)^{\alpha}\right| \leq r^{\alpha} + \frac{2\beta(1-\gamma)}{(1+\beta)\left[\frac{\alpha+j}{\alpha}\right]^{m-i}} r^{\alpha+j} \tag{24}$$

Corollary 2.10: Let the function $f(z)^{\alpha}$ defined by (3) be in the class $M_{1,0}^{\alpha,m}(\gamma,\beta)$. Then for $|z| = r < 1$

$$\left|(f(z)^{\alpha})'\right| \geq \alpha r^{\alpha-1} + \frac{2\beta(\alpha+j)(1-\gamma)}{(1+\beta)\left(\frac{\alpha+j}{\alpha}\right)^{m}} r^{\alpha+j-1} \tag{25}$$

and

$$\left|(f(z)^{\alpha})'\right| \leq \alpha r^{\alpha-1} + \frac{2\beta(\alpha+j)(1-\gamma)}{(1+\beta)\left(\frac{\alpha+j}{\alpha}\right)^{m}} r^{\alpha+j-1}. \tag{26}$$

Corollary 2.11: Let the function $f(z)^{\alpha}$ defined by (3) be in the class $M_{1,0}^{1,m}(\gamma,\beta)$, then for $|z| = r < 1$

$$\left|\frac{I^i(1,0)f(z)}{z}\right| \geq 1 - \frac{2\beta(1-\gamma)}{(1+\beta)(j+1)^{m-i}} r^j \tag{27}$$

and

$$\left|\frac{I^i(1,0)f(z)}{z}\right| \leq 1 + \frac{2\beta(1-\gamma)}{(1+\beta)(j+1)^{m-i}} r^j \tag{28}$$

Corollary 2.12: Let the function $f(z)^{\alpha}$ defined by (3) be in the class $M_{1,0}^{1,m}(\gamma,\beta)$, then for $|z| = r < 1$

$$|f(z)| \geq r^{\alpha} - \frac{2\beta(1-\gamma)}{(1+\beta)(j+1)^{m}} r^{j+1} \tag{29}$$

and

$$|f(z)| \leq r^{\alpha} + \frac{2\beta(1-\gamma)}{(1+\beta)(j+1)^{m}} r^{j+1} \tag{30}$$

Corollary 2.13: Let the function $f(z)^{\alpha}$ defined by (3) be in the class $M_{1,0}^{1,m}(\gamma,\beta)$, then for $|z| = r < 1$

$$|f'(z)| \geq 1 - \frac{2\beta(1-\gamma)}{(1+\beta)(j+1)^{m-1}} r^j \tag{31}$$

and

$$|f'(z)| \le 1 + \frac{2\beta(1-\gamma)}{(1+\beta)(j+1)^{m-1}} r^j . \tag{32}$$

Or

$$1 - \frac{2\beta(1-\gamma)}{(1+\beta)(j+1)^{m-1}} r^j \le |f'(z)| \le 1 + \frac{2\beta(1-\gamma)}{(1+\beta)(j+1)^{m-1}} r^j .$$

Modified Hadamard Products

Suppose that $f_t(z)^\alpha (t=1,2)$ be defined by;

$$f_t(z)^\alpha = z^\alpha - \sum_{k=j+1}^{\infty} a_{k,t}(\alpha) z^{\alpha+k-1} \ (a_{t,k} \ge 0;\ t=1,2) . \tag{33}$$

Then the modified Hadamard product of the function $f_t(z)^\alpha (t=1,2)$ is defined by

$$(f_1 * f_2)(z)^\alpha = z^\alpha - \sum_{k=j+1}^{\infty} a_{k,1}(\alpha) a_{k,2}(\alpha) z^k \tag{34}$$

Theorem 2.14: Let the function $f_t(z)^\alpha (t=1,2)$ defined by (33) belong to the class $M_{\lambda,l}^{\alpha,m}(\gamma,\beta)$, then, $(f_1 * f_2)(z)^\alpha \in M_{\lambda,l}^{\alpha,m}(\varepsilon,\beta)$, where

$$\varepsilon = 1 - \frac{2\beta(1-\gamma)^2}{(1+\beta)\left[\frac{1+\lambda(\alpha+j-1)+l}{1+\lambda(\alpha-1)+l}\right]^m} . \tag{35}$$

The result is the best possible.

Proof: Following Schild and Silverman [7] technique, we shall find the largest ε such that;

$$\sum_{k-j+1}^{\infty} (1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m a_{k,1}(\alpha) a_{k,2}(\alpha) \le 2\beta(1-\varepsilon) . \tag{36}$$

Since

$$\sum_{k-j+1}^{\infty} (1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m a_{k,1}(\alpha) \le 2\beta(1-\gamma) \tag{37}$$

and

$$\sum_{k-j+1}^{\infty} (1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m a_{k,2}(\alpha) \le 2\beta(1-\gamma) . \tag{38}$$

By the Cauchy – Schwarz inequality, then;

$$\sum_{k-j+1}^{\infty} \sqrt{\frac{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m}{2\beta(1-\gamma)}} a_{k,1}(\alpha) \times \sqrt{\frac{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m}{2\beta(1-\gamma)}} a_{k,2}(\alpha)$$

$$\le \left[\sum_{k=j+1}^{\infty}\left(\sqrt{\frac{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m a_{k,1}(\alpha)}{2\beta(1-\gamma)}}\right)^2\right]^{1/2} \times \left[\sum_{k=j+1}^{\infty}\left(\sqrt{\frac{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m a_{k,1}(\alpha)}{2\beta(1-\gamma)}}\right)^2\right]^{1/2}$$

It implies that;

$$\sum_{k=j+1}^{\infty} \frac{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m}{2\beta(1-\gamma)} \sqrt{a_{k,1}(\alpha) a_{k,2}(\alpha)} \le 1 . \tag{39}$$

Thus it is sufficient to show that;

$$\frac{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m a_{k,1}(\alpha)a_{k,2}(\alpha)}{2\beta(1-\varepsilon)} \le \frac{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m}{2\beta(1-\gamma)}\sqrt{a_{k,1}(\alpha)a_{k,2}(\alpha)}.$$

That is

$$\sqrt{a_{k,1}(\alpha)a_{k,2}} \le \frac{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m}{2\beta(1-\gamma)} \div \frac{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m}{2\beta(1-\varepsilon)}$$

and

$$\sqrt{a_{k,1}(\alpha)a_{k,2}(\alpha)} = \frac{1-\varepsilon}{1-\gamma}. \qquad (40)$$

Since (39) implies that;

$$\sqrt{a_{k,1}(\alpha)a_{k,2}(\alpha)} \le \frac{2\beta(1-\gamma)}{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m}. \qquad (41)$$

We need to show that;

$$\frac{2\beta(1-\gamma)}{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m} \le \frac{1-\varepsilon}{1-\gamma}.$$

That is;

$$-\varepsilon \ge \frac{2\beta(1-\gamma)^2}{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m} - 1.$$

Or equivalently,

$$\varepsilon \le 1 - \frac{2\beta(1-\gamma)^2}{(1+\beta)^2\left[\frac{1+\lambda(\alpha+j-1)+l}{1+\lambda(\alpha-1)+l}\right]^m}. \qquad (42)$$

Since the right hand of (42) isw an increasing function of k, then by letting $k=j+1$, we obtain

$$\varepsilon = 1 - \frac{2\beta(1-\gamma)^2}{(1+\beta)^2\left[\frac{1+\lambda(\alpha+j-1)+l}{1+\lambda(\alpha-1)+l}\right]^m},$$

which proves the result in theorem 2.14.The sharpness of the result of theorem 2.14 follows if we take

$$f_t(z)^\alpha = z^\alpha - \frac{2\beta(1-\gamma)}{(1+\beta)\left[\frac{1+\lambda(\alpha+j-1)+l}{1+\lambda(\alpha-1)+l}\right]^m} z^j \ (t=1,2). \qquad (43)$$

Corollary 2.15: For each function $f_1(z)^\alpha$ and $f_2(z)^\alpha$ belongs to the same class $M_{\lambda,l}^{\alpha,m}(\gamma,\beta)$, then ($(f_1 * f_2)(z)^\alpha \in M_{\lambda,l}^{\alpha,m}(\ell,1)$), where;

$$\ell = 1 - \frac{4\beta^2(1-\gamma)^2}{(1+\beta)^2\left[\frac{1+\lambda(\alpha+j-1)+l}{1+\lambda(\alpha-1)+l}\right]^m}. \qquad (44)$$

The result is sharp for the functions $f_t(z)^\alpha \ (t=1,2)$ defined by (43).

Proof: Following the same procedure as in the proof of theorem 2.14, we need to find the largest ℓ such that;

$$\sum_{k=j+1}^{\infty}\frac{\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m a_{k,1}(\alpha)a_{k,2}(\alpha)}{1-\ell}\le 1. \tag{45}$$

Now from (39) and (45), it suffices to prove that;

$$\frac{\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m a_{k,1}(\alpha)a_{k,2}(\alpha)}{(1-\ell)}\le\frac{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m\sqrt{a_{k,1}(\alpha)a_{k,2}(\alpha)}}{2\beta(1-\gamma)}$$

$$(k\ge j+1;\ j\in N).$$

Then

$$\sqrt{a_{k,1}(\alpha)a_{k,2}(\alpha)}\le\frac{(1-\ell)(1+\beta)}{2\beta(1-\gamma)}. \tag{46}$$

Using (41) and (46), we need to show that;

$$\frac{2\beta(1-\gamma)}{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m}\le\frac{(1-\ell)(1+\beta)}{2\beta(1-\gamma)},$$

Or equivalently,

$$\ell\le 1-\frac{4\beta^2(1-\gamma)^2}{(1+\beta)^2\left[\frac{1+\lambda()+l}{1+\lambda(\alpha-1)+l}\right]^m}.$$

Theorem 2.14: Let the function $f_t(z)^\alpha\,(t=1,2)$ defined by (33) be in the same class $M_{\lambda,l}^{\alpha,m}(\gamma,\beta)$, then the functions $h(z)^\alpha$ defined by;

$$h(z)^\alpha=z^\alpha-\sum_{k=j+1}^{\infty}(a_{k,1}^2+a_{k,,2}^2)z^{\alpha+k-1} \tag{47}$$

belongs to the class $M_{\lambda,l}^{\alpha,m}(\gamma,\beta)$, where;

$$\phi=1-\frac{4\beta(1-\gamma)^2}{(1+\beta)\left[\frac{1+\lambda(\alpha+j-1)+l}{1+\lambda(\alpha-1)+l}\right]^m}. \tag{48}$$

The result is sharp for the function $f_t(z)^\alpha\ (t=1,2)$ defined by (43).

Proof: With the aid of Theorem 2.1, we have that;

$$\sum_{k=j+1}^{\infty}\left\{\frac{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m}{2\beta(1-\gamma)}a_{k,1}^2\right\}\le\left\{\sum_{k=j+1}^{\infty}\frac{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m}{2\beta(1-\gamma)}a_{k,1}\right\}^2\le 1 \tag{49}$$

and

$$\sum_{k=j+1}^{\infty}\left\{\frac{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m}{2\beta(1-\gamma)}a_{k,2}^2\right\}^2\le\left\{\sum_{k=j+1}^{\infty}\frac{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m}{2\beta(1-\gamma)}a_{k,2}\right\}^2\le 1 \tag{50}$$

It follows from (49) and (50) that;

$$\sum_{k=j+1}^{\infty}\frac{1}{2}\left\{\frac{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m}{2\beta(1-\gamma)}\right\}^2(a_{k.1}^2+a_{k,2}^2)\leq 1.$$

So, we need to find the largest ϕ such that;

$$\sum_{k=j+1}^{\infty}\frac{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m}{2\beta(1-\phi)}(a_{k,1}^2+a_{k,2}^2)\leq 1.$$

That is;

$$\sum_{k=j+1}^{\infty}\frac{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m}{2\beta(1-\phi)}\leq\frac{1}{2}\left\{\frac{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m}{2\beta(1-\gamma)}\right\}^2\begin{pmatrix}k\geq j+1,\\ j\in N\end{pmatrix}.$$

Or equivalently,

$$\phi\leq 1-\frac{4\beta(1-\gamma)^2}{(1+\beta)\left[\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right]^m}.$$

This ends the proof of Theorem 2.16. For related work on coefficient inequalities, distortion properties and Hadamard product one can Castas *et al.* (2008); Opoola (1994), among others.

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Conflict of Interest

The authors hereby declare that there is no conflict of interest in the present work.

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